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The large-N limit of matrix integrals over the orthogonal group

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Abstract
The large-N limit of some matrix integrals over the orthogonal group O(N) and its relation with those pertaining to the unitary group U(N) are re-examined. It is proved that $\lim_{N\to\infty} N^{-2} \int DO \exp N \operatorname{Tr} JO$ is half the corresponding function in U(N), with a similar relation for $\lim_{N\to\infty} \int DO \exp N \operatorname{Tr}(AOBO^t)$, for A and B both symmetric or both skew symmetric.

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1. Introduction

Matrix integrals of the type

$$Z^{(G)} = \int_G D\Omega\, \mathrm{e}^{\kappa N \operatorname{Tr}(A\Omega B\Omega^{-1})}, \tag{1}$$

over a classical compact group $G = \mathrm{U}(N)$, O(N) or Sp(N), with κ being a real parameter, appear frequently in theoretical physics, from disordered systems [1–3] to 2D quantum gravity [4] and related topics. They also have a mathematical interest in connection with integrability, statistics and free probabilities. They are sometimes called matrix Bessel functions [5], or (generalized) HCIZ integrals. While the expression of Z is well known for the group U(N) [6, 7], the situation with O(N) is more subtle. The result is known for *skew-symmetric* matrices A and B [6], but its form is only partially understood for the more frequently encountered case of *symmetric* matrices, in spite of recent progress [5, 8, 9].

On the other hand, in the large-N limit, we expect things to simplify [10, 11]. It is the purpose of this work to revisit this old problem and to show that log Z has universality properties in the large-N limit, a pattern which does not seem to have been stressed enough before, at least in the physics literature (see the historical note below).

This paper is organized as follows. In section 2, we discuss the related but simpler case of the integral 'in an external field',

$$\mathcal{Z}_G = \int_G D\Omega\, \mathrm{e}^{N \operatorname{Tr}(J\Omega + J^\dagger\Omega^\dagger)} \tag{2}$$

(where the second term in the exponential will be omitted in the orthogonal case). We prove that it enjoys a universality property in the large-N limit. In section 3, we turn to integral (1), discuss its large-N limit and prove that it has a similar universality property. We also extend our integrals to cases where matrices A and B are neither symmetric (or Hermitian) nor antisymmetric.

For the sake of simplicity this paper will be focused on the case of $G = \mathrm{O}(N)$ as compared to U(N), but similar considerations apply to Sp(N).

2. The integral (2)

2.1. The basic integrals and their generating function

Let us consider the basic integral over the orthogonal group O(N)

$$\mathcal{I}_{\boldsymbol{i},\boldsymbol{j}} := \int DO\, O_{i_1 j_1} \cdots O_{i_n j_n}, \tag{3}$$

where O is an $N \times N$ orthogonal matrix and $\boldsymbol{i}$, respectively $\boldsymbol{j}$, are n-tuples of indices i respectively j.

One may easily prove that $\mathcal{I}_{\boldsymbol{i},\boldsymbol{j}}$ is non-vanishing only for even n and has then the following general structure [19]:

$$\int DO\, O_{i_1 j_1} \cdots O_{i_{2n} j_{2n}} = \sum_{p_1, p_2 \in P_{2n}} \delta_{\mathbf{i}}^{p_1} \delta_{\mathbf{j}}^{p_2} C(p_1, p_2), \tag{4}$$

where P_{2n} denotes the set of pairings in $\{1, 2, \ldots, 2n\}$,

$$\begin{aligned} &\delta_{\mathbf{i}}^{p} = 1 \qquad \text{if} \quad \forall a, b, \qquad a = p(b) \Rightarrow i_a = i_b, \\ &i \in \{1, 2, \ldots, N\} \qquad \text{and} \qquad \delta_{\mathbf{i}}^{p} = 0 \text{ otherwise,} \end{aligned} \tag{5}$$

and where the coefficients $C(p_1, p_2)$ enjoy many properties and may be determined recursively, see appendix A for a review.

The analogous basic integral for U(N) is

$$\int DU\, U_{i_1 j_1} \cdots U_{i_n j_n} U^{\dagger}_{k_1 \ell_1} \cdots U^{\dagger}_{k_n \ell_n} = \sum_{\tau,\sigma \in S_n} C([\sigma]) \prod_{a=1}^{n} \delta_{i_a \ell_{\tau(a)}} \delta_{j_a k_{\tau\sigma(a)}} \tag{6}$$

with a double sum over permutations $\sigma, \tau \in S_n$, the symmetric group, see appendix A.

Another way of encoding these formulae is to use the generating functions

$$\mathcal{Z}_{\mathrm{O}}(J, J^t; N) = \int DO\, \mathrm{e}^{N\, \mathrm{Tr}\, JO} = \mathrm{e}^{\mathcal{W}_{\mathrm{O}}(J \cdot J^t, N)}, \tag{7}$$

$$\mathcal{Z}_{\mathrm{U}}(J, J^{\dagger}; N) = \int DU\, \mathrm{e}^{N\, \mathrm{Tr}(JU + J^{\dagger} U^{\dagger})} = \mathrm{e}^{\mathcal{W}_{\mathrm{U}}(J \cdot J^{\dagger}, N)}, \tag{8}$$

where J is a generic (i.e. non-symmetric, respectively non-Hermitian) matrix. $\mathcal{Z}_{\mathrm{O}}$ depends only on traces of powers of $J \cdot J^t$, again by invariance of the integral under $J \to O_1 \cdot J \cdot O_2$. Likewise, $\mathcal{Z}_{\mathrm{U}}$ depends only on $J \cdot J^{\dagger}$.

2.2. The large-N limit and its relation to the unitary case

In the large-N limit, one may show that

$$W_{\mathrm{O}}(J \cdot J^t) = \lim_{N \to \infty} N^{-2} \mathcal{W}_{\mathrm{O}}(J \cdot J^t, N) \tag{9}$$

exists and is related to the corresponding expression for the unitary group.

Let us recall the situation in the unitary case. The generating function $\mathcal{Z}_{\mathrm{U}}$ has been studied extensively in the past. The limit $W_{\mathrm{U}}(J\cdot J^\dagger)=\lim_{N\to\infty}\mathcal{W}_{\mathrm{U}}(J\cdot J^\dagger,N)/N^2$ was shown to satisfy a partial differential equation with respect to the eigenvalues of $J\cdot J^\dagger$ [11]. In the 'strong coupling phase', an explicit expression was given [12] for the expansion of W_{U} in a series expansion in traces of powers of $J\cdot J^\dagger$

$$W_{\mathrm{U}}(J\cdot J^\dagger)=\sum_{n=1}^{\infty}\sum_{\alpha\vdash n}W_\alpha\frac{\mathrm{tr}_\alpha J\cdot J^\dagger}{\prod_p\left(\alpha_p!p^{\alpha_p}\right)} \tag{10}$$

$$W_\alpha=(-1)^n\frac{(2n+\sum\alpha_p-3)!}{(2n)!}\prod_{p=1}^{n}\left(\frac{-(2p)!}{p!(p-1)!}\right)^{\alpha_p}, \tag{11}$$

where $\alpha\vdash n$ denotes a partition of $n=\alpha_1\cdot 1+\alpha_2\cdot 2+\cdots+\alpha_n\cdot n$ and

$$\mathrm{tr}_\alpha(X):=\prod_{p=1}^{n}\left(\frac{1}{N}\,\mathrm{Tr}\,X^p\right)^{\alpha_p}. \tag{12}$$

Now we claim that W_{O} defined in (9) is

$$W_{\mathrm{O}}(J\cdot J^t)=\tfrac{1}{2}W_{\mathrm{U}}(J\cdot J^t). \tag{13}$$

Proof. We repeat the steps of [11], paying due attention to the differences between independant matrix elements in a complex Hermitian and in a real symmetric matrix. The trivial identity $\sum_j\frac{\partial^2\mathcal{Z}_{\mathrm{O}}}{\partial J_{ij}\partial J_{kj}}=N^2\delta_{ik}\mathcal{Z}_{\mathrm{O}}$ is re-expressed in terms of the eigenvalues λ_i of the real symmetric matrix $J\cdot J^t$:

$$4\lambda_i\frac{\partial^2\mathcal{Z}_{\mathrm{O}}}{\partial\lambda_i^2}+2\sum_{j\neq i}\frac{\lambda_j}{\lambda_j-\lambda_i}\left(\frac{\partial\mathcal{Z}_{\mathrm{O}}}{\partial\lambda_j}-\frac{\partial\mathcal{Z}_{\mathrm{O}}}{\partial\lambda_i}\right)+2N\frac{\partial\mathcal{Z}_{\mathrm{O}}}{\partial\lambda_i}=N^2\mathcal{Z}_{\mathrm{O}}. \tag{14}$$

Writing as above $\mathcal{Z}_{\mathrm{O}}=\mathrm{e}^{N^2W_{\mathrm{O}}}$ and dropping subdominant terms in the large-N limit, with W_{O} and $W_i:=N\partial W_{\mathrm{O}}/\partial\lambda_i$ of order 1, we obtain

$$4\lambda_iW_i^2+2W_i+\frac{1}{N}\sum_{j\neq i}\frac{2\lambda_i}{\lambda_j-\lambda_i}(W_j-W_i)=0, \tag{15}$$

which is precisely the equation satisfied by $\frac{1}{2}W_{\mathrm{U}}$ in [11]. This, supplemented by appropriate boundary conditions, suffices to complete the proof of (13). □

As noted elsewhere [7, 13] it is appropriate to expand W on 'free' (or 'non-crossing') cumulants [7, 14, 15]

$$\psi_q(X):=-\sum_{\substack{\alpha_1,\ldots,\alpha_q\geqslant 0\\ \sum_i i\alpha_i=q}}\frac{(q+\sum_i\alpha_i-2)!}{(q-1)!}\prod_i\frac{\left(-\frac{1}{N}\,\mathrm{Tr}\,X^i\right)^{\alpha_i}}{\alpha_i!}. \tag{16}$$

Then

$$W_{\mathrm{O}}(J\cdot J^t)=\tfrac{1}{2}\psi_1(J\cdot J^t)-\tfrac{1}{4}\psi_2(J\cdot J^t)+\tfrac{1}{3}\psi_3(J\cdot J^t)-\tfrac{1}{8}(5\psi_4(J.J^t)+\psi_2^2(J\cdot J^t))+\cdots, \tag{17}$$

where the only occurrence of ψ_1 is in the first term.

3. The generalized HCIZ integral

3.1. Notations. Review of known results

Let us first recall the well-known results. If the matrices A and B in (1) are in the Lie algebra of the group G, namely are real skew-symmetric, respectively anti-Hermitian, for $G = \mathrm{O}(N)$, respectively $\mathrm{U}(N)$, the exact expression of $Z^{(G)}$ is known from the work of Harish-Chandra [6]. To make the formulae quite explicit, we take A and B in the Cartan torus, i.e. of a diagonal or block-diagonal form [16]

$$A = \mathrm{diag}(a_i)_{i=1,\ldots,N} \qquad \text{for} \quad \mathrm{U}(N) \tag{18}$$

and

$$A = \begin{cases} \mathrm{diag}\left(\begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}_{i=1,\ldots,m}\right) & \text{for} \quad O(2m) \\ \mathrm{diag}\left(\begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}_{i=1,\ldots,m}, 0\right) & \text{for} \quad O(2m+1) \end{cases} \tag{19}$$

and likewise for B. We assume that all the a's are distinct and likewise for the b's. Then the integral (1) reads

$$Z^{(G)} = \text{const.} \frac{\det(\mathcal{M}_G)}{\Delta_G(a)\Delta_G(b)}, \tag{20}$$

where

$$\begin{aligned} \mathcal{M}_G &= \begin{cases} (\mathrm{e}^{\kappa N a_i b_j})_{1\leqslant i,j\leqslant N} \\ (\mathrm{e}^{2\kappa N a_i b_j} + \mathrm{e}^{-2\kappa N a_i b_j})_{1\leqslant i,j\leqslant m} \\ (\mathrm{e}^{2\kappa N a_i b_j} - \mathrm{e}^{-2\kappa N a_i b_j})_{1\leqslant i,j\leqslant m}, \end{cases} \\ \Delta_G(a) &= \begin{cases} \prod_{i<j}(a_i - a_j) & \text{if} \quad G = \mathrm{U}(N) \\ \prod_{1\leqslant i<j\leqslant m}\left(a_i^2 - a_j^2\right) & \text{if} \quad G = \mathrm{O}(2m) \\ \prod_{1\leqslant i<j\leqslant m}\left(a_i^2 - a_j^2\right)\prod_{i=1}^m a_i & \text{if} \quad G = \mathrm{O}(2m+1). \end{cases} \end{aligned} \tag{21}$$

For convenience, $\Delta_{\mathrm{U}}(a)$ will be abbreviated into $\Delta(A) = \prod_{i<j}(a_i - a_j)$, the usual Vandermonde determinant.

3.2. The large-N limit

We now consider

$$F^{(\mathrm{O})}(A,B) := \lim N^{-2}\log\int DO\,\mathrm{e}^{N\,\mathrm{Tr}\,AOBO'}, \tag{22}$$

$$F^{(\mathrm{U})}(A,B) := \lim N^{-2}\log\int DU\,\mathrm{e}^{\mathbf{2}N\,\mathrm{Tr}\,AUBU^\dagger}. \tag{23}$$

Note the factor 2 introduced for convenience in the latter exponential. We claim that, for A and B both symmetric or skew-symmetric,

$$F^{(\mathrm{O})}(A,B) = \tfrac{1}{2}F^{(\mathrm{U})}(A,B). \tag{24}$$

3.2.1. The skew-symmetric case. For A and B both skew-symmetric, the limit of (20) is easy to evaluate. Assuming without loss of generality that all a's and b's are positive, we find that $(\mathcal{M}_{\mathrm{O}})_{ij} \sim \mathrm{e}^{2Na_ib_j}$, thus $Z^{(\mathrm{O})} \sim \det(\mathrm{e}^{2Na_ib_j})/\Delta_{\mathrm{O}}(a)\Delta_{\mathrm{O}}(b)$. On the other hand, the real matrices A and B of the form (19) may also be regarded as anti-Hermitian, with eigenvalues $A_j = \pm \mathrm{i}a_j$, $j = 1, \ldots, m$ (supplemented by 0 if $N = 2m + 1$). An easy computation gives $\Delta(A) = (\Delta_{\mathrm{O}}(a))^2$ up to a sign. The matrix $\mathcal{M}_{\mathrm{U}} = (\mathrm{e}^{2NA_iB_j})_{1\leqslant i,j\leqslant N} \approx (\mathrm{e}^{2Na_ib_j})_{1\leqslant i,j\leqslant m} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, as $N \to \infty$. Hence $\det \mathcal{M}_{\mathrm{U}} \sim (\det \mathcal{M}_{\mathrm{O}})^2$. Thus, the U($N$) integration for the pair (A, B) yields, according to (21), and up to an overall factor,

$$Z^{(\mathrm{U})}(A, B) = \frac{\det(\mathrm{e}^{2NA_iB_j})}{\Delta(A)\Delta(B)} \sim \left(\frac{(\det(\mathrm{e}^{2Na_ib_j})_{1\leqslant i,j\leqslant m}}{\Delta_{\mathrm{O}}(a)\Delta_{\mathrm{O}}(b)}\right)^2 = \text{const.}\,(Z^{(\mathrm{O})(A,B)})^2 \tag{25}$$

in accordance with (24).

3.2.2. The symmetric case. We now turn to the more challenging case where both A and B are real symmetric. We may suppose that A and B are in a diagonal form $A = \mathrm{diag}(a_i)$, $B = \mathrm{diag}(b_i)$, and assume that all a's and all b's are distinct. In that case, we shall resort to (an infinite set of) differential equations, in a way similar to the discussion of section 2.

In a recent work, Bergère and Eynard [9] have introduced the following integrals over the compact group G:

$$M_{ij} = \int_G \mathrm{d}\Omega\, \Omega_{ij}\Omega^{-1}_{ji}\, \mathrm{e}^{\kappa N \operatorname{Tr} A\Omega B\Omega^{-1}}, \tag{26}$$

which may be regarded as particular two-point correlation functions associated with the partition function $Z^{(G)}$ of (1). The latter is recovered from M_{ij} by a summation over i or j

$$Z^{(G)} = \sum_i M_{ij}, \forall j, \qquad Z^{(G)} = \sum_j M_{ij}, \forall i. \tag{27}$$

As shown in [9], M_{ij} satisfy the following set of differential equations:

$$\sum_j K_{ij}M_{jk} = \kappa N M_{ik}b_k \tag{28}$$

with no summation over k in the rhs. Here K refers to the matrix differential operator

$$K_{ii} = \frac{\partial}{\partial a_i} + \frac{\beta}{2}\sum_{j\neq i}\frac{1}{a_i - a_j} \qquad \text{and for} \quad i \neq j, \qquad K_{ij} = -\frac{\beta}{2}\frac{1}{a_i - a_j}. \tag{29}$$

(We make use of Dyson's label $\beta = 1, 2$ for $G = \mathrm{O}(N), \mathrm{U}(N)$, respectively.)

Now, by a repeated application of the operator K on M, we find for any positive integer p that $\sum_j (K^p)_{ij}M_{jk} = N^p M_{ik}b_k^p$; hence after summation over i and k,

$$\sum_{i,j}(K^p)_{ij}Z^{(G)} = Z^{(G)}\sum_k(\kappa N b_k)^p. \tag{30}$$

The differential operator $D_p := \sum_{1\leqslant i,j\leqslant N}(K^p)_{ij}$ has thus the property that

$$D_p Z^{(G)} = (\kappa N)^p \operatorname{Tr} B^p Z^{(G)}. \tag{31}$$

Thus far, the discussion holds for any finite value of N. Now take the large-N limit with the ansatz $Z^{(G)} = \mathrm{e}^{N^2F^{(G)}}$. Equation (31) reduces in that limit to

$$\sum_i\left(\frac{N}{\kappa}\frac{\partial F^{(G)}}{\partial a_i} + \frac{\beta}{2\kappa N}\sum_{j\neq i}\frac{1}{a_i - a_j}\right)^p = \operatorname{Tr} B^p \tag{32}$$

with $N\frac{\partial F}{\partial a_i}$ of order 1, as in section 2, and all other terms resulting from further application of ∂/∂_{a_i} over $F^{(G)}$ suppressed by inverse powers of N.

These equations had been obtained in [7] in the case of $\mathrm{U}(N)$ $(\beta = 2)$ from the explicit form of $Z^{(\mathrm{U})}$ and shown to determine recursively the expansion of $F^{(\mathrm{U})}$ in traces of powers of A and B. Comparing the orthogonal $(\kappa = \beta = 1)$ and unitary $(\kappa = \beta = 2)$ cases, it is clear in (32) that $2F^{(\mathrm{O})}(A, B)$ satisfies the same set of equations as $F^{(\mathrm{U})}(A, B)$, thus vindicating (24).

3.3. The generic case

Although it does not look very natural in view of their symmetries, one may extend the integrals $Z^{(\mathrm{U})}$ and $Z^{(\mathrm{O})}$ to the case of generic complex (non-Hermitian), respectively real (neither symmetric nor skew-symmetric), matrices A and B. If we insist on having real quantities in the exponential, the unitary integral that we consider reads

$$Z^{(\mathrm{U})} = \int \mathrm{d}U\, \mathrm{e}^{\mathrm{Tr}(AUBU^\dagger + A^\dagger U B^\dagger U^\dagger)} \tag{33}$$

(and one recovers the factor 2 introduced in (23) for A and B Hermitian). In parallel the more general orthogonal integral reads

$$Z^{(\mathrm{O})} = \int \mathrm{d}O\, \mathrm{e}^{\mathrm{Tr}\, AOBO^t} = \int \mathrm{d}O\, \mathrm{e}^{\mathrm{Tr}\, A^t OB^t O^t}. \tag{34}$$

The functions $Z^{(\mathrm{U})}$ and $F^{(\mathrm{U})}$ have now expansions in traces of products of A and $A^\dagger$ (or A^t) and of B and $B^\dagger$ (or B^t), with an equal number of daggers (respectively transpositions) appearing on A and B. We can no longer rely on the diagonal form of A and B (a generic real, respectively complex matrix is *not* diagonalized by an orthogonal, respectively unitary, matrix) and there are no longer differential equations in these eigenvalues satisfied by Z or F. Still, there is some evidence that universality holds again. By expanding the exponentials and by making use of the explicit integrals (4), (6), (see also appendix A), we have checked that, up to the fourth order, for A and B real

$$F^{(\mathrm{O})}(A, A^t; B, B^t) = \tfrac{1}{2} F^{(\mathrm{U})}(A, A^\dagger = A^t; B, B^\dagger = B^t). \tag{35}$$

If we write the expansion of F in powers of A (and B) as $F = \sum_{n=1} F_n$, we find

$$F_1^{(\mathrm{U})} = \psi_1(A)\psi_1(B) + \psi_1(A^\dagger)\psi_1(B^\dagger), \tag{36}$$

$$F_2^{(\mathrm{U})} = \tfrac{1}{2}(\psi_2(A)\psi_2(B) + 2\psi_{2t}(A, A^\dagger)\psi_{2t}(B, B^\dagger) + \psi_2(A^\dagger)\psi_2(B^\dagger)), \tag{37}$$

$$F_3^{(\mathrm{U})} = \tfrac{1}{3}(\psi_3(A)\psi_3(B) + 3\psi_{3t}(A, A^\dagger)\psi_{3t}(B, B^\dagger) + (A \to A^\dagger, B \to B^\dagger)), \tag{38}$$

$$\begin{aligned} F_4^{(\mathrm{U})} = \tfrac{1}{4}\big(&\psi_4(A)\psi_4(B) + 4\psi_{4t}(A, A^\dagger)\psi_{4t}(B, B^\dagger) + 2\psi_{4tt}(A, A^\dagger)\psi_{4tt}(B, B^\dagger) \\ &+ \psi_{4t|t}(A, A^\dagger)\psi_{4t|t}(B, B^\dagger) - \psi_2^2(A)\psi_2^2(B) - \psi_{2t}^2(A, A^\dagger)\psi_{2t}^2(B, B^\dagger) \\ &- \psi_{2t}^2(A, A^\dagger)\psi_2(B)\psi_2(B^\dagger) - \psi_2(A)\psi_2(A^\dagger)\psi_{2t}^2(B, B^\dagger) \\ &- 4\psi_2(A)\psi_{2t}^2(A, A^\dagger)\psi_2(B)\psi_{2t}^2(B, B^\dagger) + (A \to A^\dagger, B \to B^\dagger)\big), \end{aligned} \tag{39}$$

where the ψ_n are the free cumulants defined above and the ψ_{nt} are 'polarized' versions of the latter, involving A and $A^\dagger$ (or A^t), see appendix B for explicit expressions.

4. Concluding remarks

- It should be stressed that the equality (24) can be true only asymptotically as $N \to \infty$. Indeed the exact result [6] for A and B skew-symmetric as well as what is known for A and B both symmetric [8] clearly indicate that it does not hold for finite N.
- The differential operator D_p considered above is interesting in its own right. Consider the differential operator $\hat{D}_p(\partial/\partial A)$ such that

$$\hat{D}_p(\partial/\partial A)\,\mathrm{e}^{N\,\mathrm{Tr}\,AB} = N^p\,\mathrm{Tr}\,B^p\,\mathrm{e}^{\mathrm{Tr}\,AB}. \tag{40}$$

For Hermitian matrices, for which all matrix elements A_{ij} may be regarded as independent, one may write

$$D_p(\partial/\partial A) = \mathrm{Tr}\left(\frac{\partial}{\partial A}\right)^p := \sum_{i_1,\cdots i_p} \frac{\partial}{\partial A_{i_1 i_2}} \frac{\partial}{\partial A_{i_2 i_3}} \cdots \frac{\partial}{\partial A_{i_p i_1}}, \tag{41}$$

while in the case of symmetric matrices, the general expression involves some combinatorial factors. The above property (40) suffices to define $\hat{D}_p$ on any (differentiable) function of A, by the Fourier transform.

Now let $\hat{D}_p$ act on functions $f(A)$ invariant upon $A \to \Omega A \Omega^{-1}$. Then $\hat{D}_p$ reduces to a differential operator D_p on the eigenvalues a_i of A. As we have seen above, $D_p = \sum_{ij}(K^p)_{ij}$, but it would seem desirable to have a more direct construction of that basic operator. In the case of $G = \mathrm{U}(N)$, one has the elegant form [7]

$$D_p = \frac{1}{\Delta(A)} \sum_i \left(\frac{\partial}{\partial a_i}\right)^p \Delta(A). \tag{42}$$

This result, however, makes use of the explicit form (20) of $Z^{(\mathrm{U})}$, and there is no counterpart for $G = \mathrm{O}(N)$. Thus the question is: can one derive the expression (42) of D_p from that (41) of $\hat{D}_p$? Curiously, what looks like an innocent exercise of calculus turns out to be non-trivial, even for $G = \mathrm{U}(N)$.
- In view of the similarity between (13) and (24), on one hand, and of our (partial) results and conjecture on the 'generic' case, on the other, it would be nice to have a general, intuitive argument why these universality properties hold. Heuristically, the overall factor $\frac{1}{2}$ in (13) and (24) just reflects the ratio of numbers of degrees of freedom in the two cases: there are $N(N-1)/2 \sim N^2/2$ real parameters in an orthogonal matrix and N^2 in a unitary one. But why is the function of A and B universal?
- *Diagrammatics?* A diagrammatic expansion exists for $F^{(\mathrm{U})}$ [13], using the functional $\mathcal{W}$ of section 2, and this matches a combinatorial expansion [17]. Repeating the argument in the real orthogonal case leads to a much less transparent result, however, and does not seem to yield a simple derivation of (24) based on (13).
- *A heuristic argument.* Our result (24) for real symmetric versus complex Hermitian matrices should also be related to a similar relation between partition functions of two-matrix models. It is 'well known' that integrals over two real symmetric, respectively two complex Hermitian matrices with arbitrary polynomial potentials V and W,

$$Z_{2\mathrm{RS}} = \int_{\substack{\text{real}\\ \text{symmetric}}} \mathrm{d}A\,\mathrm{d}B\,\mathrm{e}^{-N\,\mathrm{Tr}(V(A)+W(B)-AB)} \sim \mathrm{e}^{-N^2 F_{2\mathrm{RS}}}, \tag{43}$$

$$Z_{2\mathrm{CH}} = \int_{\substack{\text{complex}\\ \text{Hermitian}}} \mathrm{d}A\,\mathrm{d}B\,\mathrm{e}^{-2N\,\mathrm{Tr}(V(A)+W(B)-AB)} \sim \mathrm{e}^{-N^2 F_{2\mathrm{CH}}}, \tag{44}$$

are such that for large N

$$F_{2\mathrm{CH}} = 2F_{2\mathrm{RS}}. \tag{45}$$

(Note once again the factor 2 in front of the 'action' of the Hermitian case.)
For one-matrix integrals, this is a classical result, following from the saddle point approximation [14] or from the orthogonal polynomial approach [18]. It may also be derived from the diagrammatics: Feynman diagrams for real symmetric matrices in the large-N limit are the same as those of the Hermitian integral, up to factors of 2 coming from the possible twists of the double lines of their propagators. This diagrammatic argument is expected to extend to the two-matrix integrals (44), justifying the claim (45).

On the other hand, if we diagonalize the matrices $A = \mathrm{diag}(a_i)$ and $B = \mathrm{diag}(b_i)$, we see that (44) reduces to

$$Z_{2RS} = \int \mathrm{d}\mathbf{a}\, \mathrm{d}\mathbf{b} |\Delta(a)\Delta(b)|\, \mathrm{e}^{-N\sum_i (V(a_i))+W(b_i))} \int DO\, \mathrm{e}^{N\,\mathrm{Tr}\, AOBO^t}, \tag{46}$$

$$Z_{2CH} = \int \mathrm{d}\mathbf{a}\, \mathrm{d}\mathbf{b} (\Delta(a)\Delta(b))^2\, \mathrm{e}^{-2N\sum_i (V(a_i))+W(b_i))} \int DU\, \mathrm{e}^{N\,\mathrm{Tr}\, AUBU^t}. \tag{47}$$

Finally, if we imagine that the latter integrals over the eigenvalues are dominated in the large-N limit by a saddle point configuration, we see that the scaling (24) of the angular part is *consistent* with the scaling (45) of the full integral. Obviously a more rigorous version of this heuristic argument would be desirable.

- *Historical remarks*. As far as we know, the property (13) had never been observed before. On the other hand, property (24) has a richer history. It seems to have been first observed in the case where A or B is of finite *rank* in [2], and then repeatedly used in the physics literature [22, 21]. This was later proved in a rigorous way in [17]. In [20], this is extended to the case where the rank is $o(N)$. Indeed for a finite rank of A, say, only terms with a single trace of some power of A dominate, and the expression of $F(A, B)$ is known to be given by $\sum_{n\geqslant 1} \frac{1}{n}\frac{1}{N}\,\mathrm{Tr}\, A^n \psi_n(B)$ for the unitary group [7].

  Following a totally different approach, Guionnet and Zeitouni [23] have proved rigorously the existence of the free energies $F^{(\mathrm{U})}$ and $F^{(\mathrm{O})}$ (for A and B symmetric) in the large limit, and have established that they solve the flow equation proposed by Matytsin [24]. A by-product of their discussion is the explicit β dependence of the free energy and the resulting universality property (24). This has been made more explicit in the recent paper [25]. These papers also cover the case of the symplectic group ($\beta = 4$).

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Appendix A. More details on the 'basic integrals'

In this appendix, we recall well-known results [10] on the integral (3). Equivalently we may consider

$$\mathcal{I}(\boldsymbol{u}, \boldsymbol{v}) = \int DO \prod_{a=1}^{n} (\boldsymbol{u}_a \cdot O\boldsymbol{v}_a), \tag{A.1}$$

where $\boldsymbol{u}_a$ and $\boldsymbol{v}_a$, $a = 1, \ldots, n$, are vectors of $\mathbb{R}^N$. The integral $\mathcal{I}(\boldsymbol{u}, \boldsymbol{v})$ is linear in each $\boldsymbol{u}_a$ and each $\boldsymbol{v}_a$, and is invariant under a global rotation of all $\boldsymbol{u}$'s or of all $\boldsymbol{v}$'s: $\boldsymbol{u}_a \to O_1\boldsymbol{u}_a$, $\boldsymbol{v}_a \to O_2\boldsymbol{v}_a$, since this may be absorbed by the change of integration variable $O_1^t O O_2 \to O$ in accordance with the invariance of the Haar measure DO. If $N > n$ the completely antisymmetric tensor ϵ cannot be used to build invariants. Hence $\mathcal{I}(\boldsymbol{u}, \boldsymbol{v})$ is only a function of the invariants $\boldsymbol{u}_a.\boldsymbol{u}_b$, $\boldsymbol{v}_a.\boldsymbol{v}_b$ and by linearity must be of the form

$$\mathcal{I}(\boldsymbol{u}, \boldsymbol{v}) = \sum_{p_1, p_2} C(p_1, p_2) \prod (\boldsymbol{u}_a \cdot \boldsymbol{u}_{p_1(a)}) \prod (\boldsymbol{v}_b \cdot \boldsymbol{v}_{p_2(b)}), \qquad (A.2)$$

a sum over all possible *pairings* of the indices $a = 1, \ldots, n$, $b = 1, \ldots, n$; this shows that $\mathcal{I}$ vanishes for n odd. In the following we change $n \to 2n$ and denote P_{2n} the set of all pairings of $\{1, 2, \ldots, 2n\}$, with $|P_{2n}| = (2n-1)!!$.

Then the general expression of $\int DO\, O_{i_1 j_1} \cdots O_{i_{2n} j_{2n}}$ is indeed of the form (4). The coefficients $C(p_1, p_2)$ may be determined recursively, but let us first point some general features.

(i) Regard now p_1 and p_2 as permutations of S_{2n}, both in the class $[2^n]$ of permutations made of n 2-cycles (transpositions). Represent a typical term in the rhs. of (4) by a set of disjoint chain loops $i_a - j_a - j_{p_2(a)} - i_{p_2(a)} - i_{p_1.p_2(a)} - \cdots$ (these are the loop diagrams of [10]). The coefficients $C(p_1, p_2)$ are thus only functions of the product $p_1.p_2$, and in fact functions only of the *class* in S_{2n} of that product. Indeed if all i and j indices are relabelled through the same permutation $\pi \in S_{2n}$, $i_a \to i'_a = i_{\pi(a)}$, $j_a \to j'_a = j_{\pi(a)}$, $a = 1, \ldots, 2n$, the integrand is preserved and $p_s \to p'_s = \pi^{-1} \cdot p_s \cdot \pi$, for $s = 1, 2$, hence $p_1 \cdot p_2 \to \pi^{-1} \cdot p_1 \cdot p_2 \cdot \pi$ and $C(p_1 \cdot p_2)$ must depend only on the class $[p_1 \cdot p_2]$.

(ii) For p_1 and $p_2 \in [2^n]$, their product $p_1 \cdot p_2$ is the product of two permutations of S_n acting on two disjoints subsets of n elements of $\{1, 2, \ldots, 2n\}$, both in the same class of S_n, $p_1.p_2 = \sigma.\sigma'$ with $[\sigma] = [\sigma']$ [20]. The class $[p_1.p_2]$ of $p_1.p_2$ is completely specified by $[\sigma]$; hence we may write the coefficients as $C(p_1, p_2) = C([\sigma])$.

Proof. To any cycle α of $p_1.p_2$, $\{a, p_1.p_2(a), (p_1.p_2)^2(a), \ldots, (p_1.p_2)^r(a)\}$, one may associate another one $\{p_1(a), p_1.p_2.p_1(a), (p_1.p_2)^2 p_1(a), \ldots, (p_1.p_2)^r p_1(a)\}$, which is obviously of the same length and which acts on distinct elements. Thus $p_1.p_2 = \sigma.\sigma'$, where σ and σ' acting on distinct elements of $\{1, 2, \ldots, 2n\}$ may be regarded as in the same class of S_n. Moreover the class $[p_1 \cdot p_2]$, i.e. the cycle structure of $p_1 \cdot p_2$ is obviously given by that of $[\sigma] = [\sigma']$. □

The coefficients C are then determined recursively. Noting that by contracting the last two j indices one constructs $O_{i_{n-1} j_{n-1}} O_{i_n j_n} \delta_{j_{n-1} j_n} = (O \cdot O^t)_{i_{n-1} i_n} = \delta_{i_{n-1} i_n}$, and one gets a (strongly overdetermined) system of equations relating the C's of order n to those of order $n-1$ [10]. Explicit although fairly complicated solutions have been given [19].

The first coefficients read explicitly

$$n = 1 \quad C[1] = \frac{1}{N}$$

$$n = 2 \quad C[2] = \frac{-1}{N(N-1)(N+2)}, \qquad C[1,1] = \frac{N+1}{N(N-1)(N+2)}$$

$$n = 3 \quad C[3] = \frac{2}{(N-2)(N-1)N(N+2)(N+4)}, \quad C[1,2] = \frac{-1}{(N-2)(N-1)N(N+4)},$$

$$C[1^3] = \frac{N^2+3N-2}{(N-2)(N-1)N(N+2)(N+4)}$$

$$n=4 \quad C[4]=\frac{-(5N+6)}{(N-3)(N-2)(N-1)N(N+1)(N+2)(N+4)(N+6)},$$
$$C[1,3]=\frac{2}{(N-3)(N-2)(N-1)(N+1)(N+2)(N+6)},$$
$$C[2^2]=\frac{N^2+5N+18}{(N-3)(N-2)(N-1)N(N+1)(N+2)(N+4)(N+6)},$$
$$C[1^2,2]=\frac{-(N^3+6N^2+3N-6)}{(N-3)(N-2)(N-1)N(N+1)(N+2)(N+4)(N+6)},$$
$$C[1^4]=\frac{(N+3)\left(N^2+6N+1\right)}{(N-3)(N-1)N(N+1)(N+2)(N+4)(N+6)}.$$

The analogous basic integrals in U(N) are more widely known, see (6). One may actually give an explicit form to the $C([\sigma.\tau])$, namely

$$\int DU U_{i_1 j_1}\cdots U_{i_n j_n} U^\dagger_{k_1\ell_1}\cdots U^\dagger_{k_n\ell_n} = \sum_{\tau,\sigma\in S_n}\ \sum_{\substack{Y\,\text{Young diagr.}\\ |Y|=n}} \frac{(\chi^{(\lambda)}(1))^2\chi^{(\lambda)}([\sigma])}{n!^2 s_\lambda(I)} \prod_{a=1}^n \delta_{i_a\ell_{\tau(a)}}\delta_{j_a k_{\tau\sigma(a)}},$$

where $\chi^{(\lambda)}([\sigma])$ is the character of the symmetric group S_n associated with the Young diagram Y, hence a function of the class $[\sigma]$ of σ; $\chi^{(\lambda)}(1)$ is thus the dimension of that representation; $s_\lambda(X)$ is the character of the linear group GL(N) associated with the Young diagram Y, i.e. a Schur function when expressed in terms of the eigenvalues of X; $s_\lambda(I)$ is thus the dimension of that representation.

The first coefficients read explicitly

$$n=1 \quad C[1]=\frac{1}{N}$$
$$n=2 \quad C[2]=-\frac{1}{(N-1)N(N+1)}, \qquad C[1,1]=\frac{1}{(N-1)(N+1)}$$
$$n=3 \quad C[3]=\frac{2}{(N-2)(N-1)N(N+1)(N+2)},$$
$$C[2,1]=-\frac{1}{(N-2)(N-1)(N+1)(N+2)},$$
$$C[1^3]=\frac{N^2-2}{(N-2)(N-1)N(N+1)(N+2)}$$
$$n=4 \quad C[4]=-\frac{5}{(N-3)(N-2)(N-1)N(N+1)(N+2)(N+3)},$$
$$C[3,1]=\frac{2N^2-3}{(N-3)(N-2)(N-1)N^2(N+1)(N+2)(N+3)},$$
$$C[2^2]=\frac{N^2+6}{(N-3)(N-2)(N-1)N^2(N+1)(N+2)(N+3)},$$
$$C[2,1^2]=-\frac{1}{(N-3)(N-1)N(N+1)(N+3)},$$
$$C[1^4]=\frac{N^4-8N^2+6}{(N-3)(N-2)(N-1)N^2(N+1)(N+2)(N+3)}.$$

Appendix B. Free (non-crossing) cumulants

Note that in this appendix, we make use of a different notation for normalized traces $\mathrm{tr}\, X := \frac{1}{N}\,\mathrm{Tr}\, X$, $\mathrm{Tr}\, X$ the usual trace, thus $\mathrm{tr}\, I = 1$.

For convenience, we list here the first free cumulants of A in terms of the $\phi_p(A) = \mathrm{tr}\, A^p = \frac{1}{N}\,\mathrm{Tr}\, A^p$ together with the mixed ones, involving traces of products A and $A^\dagger$:

$$\psi_1(A) = \mathrm{tr}\, A,$$
$$\psi_2(A) = \mathrm{tr}\, A^2 - (\mathrm{tr}\, A)^2,$$
$$\psi_3(A) = \mathrm{tr}\, A^3 - 3\,\mathrm{tr}\, A\,\mathrm{tr}\, A^2 + 2(\mathrm{tr}\, A)^3,$$
$$\psi_4(A) = \mathrm{tr}\, A^4 - 4\,\mathrm{tr}\, A\,\mathrm{tr}\, A^3 - 2(\mathrm{tr}\, A^2)^2 + 10(\mathrm{tr}\, A)^2\,\mathrm{tr}\, A^2 - 5(\mathrm{tr}\, A)^4,$$
$$\psi_{2t}(A, A^\dagger) = \mathrm{tr}(AA^\dagger) - \mathrm{tr}\, A\,\mathrm{tr}\, A^\dagger,$$
$$\psi_{3t}(A, A^\dagger) = \mathrm{tr}(A^2A^\dagger) - \mathrm{tr}\, A^\dagger\,\mathrm{tr}(A^2) - 2\,\mathrm{tr}\, A\,\mathrm{tr}(AA^\dagger) + 2(\mathrm{tr}\, A)^2\,\mathrm{tr}\, A^\dagger,$$
$$\begin{aligned}\psi_{4t}(A, A^\dagger) = {} & \mathrm{tr}(A^3A^\dagger) - \mathrm{tr}\, A^\dagger\,\mathrm{tr}\, A^3 - 3\,\mathrm{tr}\, A\,\mathrm{tr}(A^2A^\dagger)) - 2\,\mathrm{tr}\, A^2\,\mathrm{tr}(AA^\dagger) + 5\,\mathrm{tr}\, A\,\mathrm{tr}\, A^\dagger\,\mathrm{tr}(A^2)\\ & + 5(\mathrm{tr}\, A)^2\,\mathrm{tr}(AA^\dagger)) - 5\,\mathrm{tr}^3 A\,\mathrm{tr}\, A^\dagger,\end{aligned}$$
$$\begin{aligned}\psi_{4tt}(A, A^\dagger) = {} & \mathrm{tr}(A^2A^{\dagger 2}) - 2\,\mathrm{tr}\, A^\dagger\,\mathrm{tr}(A^2A^\dagger) - 2\,\mathrm{tr}\, A\,\mathrm{tr}(AA^{\dagger 2}) - \mathrm{tr}(A^2)\,\mathrm{tr}(A^{\dagger 2}) - (\mathrm{tr}(AA^\dagger))^2\\ & + 2(\mathrm{tr}\, A)^2\,\mathrm{tr}(A^{\dagger 2}) + 2(\mathrm{tr}\, A^\dagger)^2\,\mathrm{tr}(A^2) + 6\,\mathrm{tr}\, A\mathrm{tr}\, A^\dagger\,\mathrm{tr}(AA^\dagger) - 5\,\mathrm{tr}^2 A\,\mathrm{tr}^2 A^\dagger,\end{aligned}$$
$$\begin{aligned}\psi_{4t-t}(A, A^\dagger) = {} & \mathrm{tr}((AA^\dagger)^2) - 2\,\mathrm{tr}\, A^\dagger\,\mathrm{tr}(A^2A^\dagger) - 2\,\mathrm{tr}\, A\,\mathrm{tr}(AA^{\dagger 2}) - 2(\mathrm{tr}(AA^\dagger))^2\\ & + (\mathrm{tr}\, A^\dagger)^2\,\mathrm{tr}\, A^2 + (\mathrm{tr}\, A)^2\,\mathrm{tr}\, A^{\dagger 2} + 8\,\mathrm{tr}\, A\mathrm{tr}\, A^\dagger\,\mathrm{tr}(AA^\dagger)) - 5(\mathrm{tr}\, A)^2(\mathrm{tr}\, A^\dagger)^2.\end{aligned}$$

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